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ADAPTIVE METHODS AND ERROR ESTIMATION FOR ELLIPTIC
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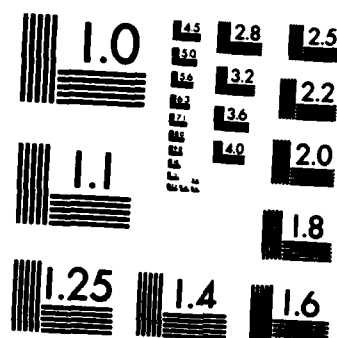
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ADAPTIVE METHODS AND ERROR ESTIMATION FOR
ELLIPTIC PROBLEMS OF STRUCTURAL MECHANICS

by

I. Babuška, A. Miller and M. Vogelius

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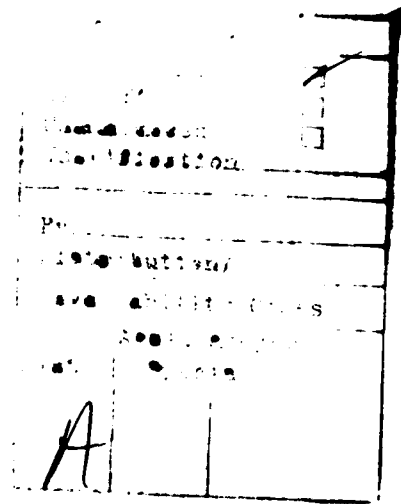
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<p>— The authors</p> <p>20. ABSTRACT (Continue on reverse side if necessary and identify by block number) We develop a class of feedback finite element methods for a one dimensional boundary value problem, and we state some mathematical results about their optimality with respect to various performance measures. Analogous ideas and results are discussed for a two dimensional problem from plane elasticity. A post-processing technique aimed at computing the value of a functional with high accuracy is also described and illustrated by an example.</p>		



at a particular point) with a prescribed error-tolerance.

In Section 2 we discuss for illustration a simple one dimensional model problem. We introduce the notion of a feedback finite element method and define adaptivity to mean optimality with respect to a given performance measure. The Theorems 2.2, 2.3 and 2.4 verify adaptivity for some specific feedback finite element methods.

Section 3 contains a discussion of the adaptive features of the FEARS program (Finite Element Adaptive Research Solver) developed at the University of Maryland. The numerical example is the computation of the stress state of a cracked panel.

In Section 4 we address the question of adaptivity in a case where the aim of the computation is to find the maximal stress of a twisted bar.

2. The One Dimensional Problem. Consider the model two point boundary value problem on $I = (0,1)$

$$(2.1a) \quad Lu \equiv -\frac{d}{dx} \left(a(x) \frac{du}{dx} \right) + b(x)u = f(x), \quad x \in I$$

$$(2.1b) \quad u(0) = u(1) = 0$$

where $a, b \in L_\infty(I)$ with

$$(2.2a) \quad 0 < \alpha_1 \leq a(x) \leq \alpha_2 < \infty$$

$$(2.2b) \quad 0 \leq b(x) \leq \alpha_2 < \infty$$

and

$$(2.2c) \quad f \in H^{-1}(I)$$

The solution $u \in H^1(I)$ is to be understood in the usual weak form

$$B(u, v) = \int_0^1 f v dx \quad \forall v \in H^1(I)$$

where B denotes the bilinear form

$$(2.3) \quad B(u, v) = \int_0^1 \left(a \frac{du}{dx} \frac{dv}{dx} + buv \right) dx \quad u, v \in H^1(I).$$

The finite element solution is introduced in the standard way. Let Δ be an arbitrary mesh

$$\Delta : = 0 = x_0^\Delta < x_1^\Delta < \dots < x_{N(\Delta)}^\Delta = 1$$

and let $I_j^\Delta = (x_{j-1}^\Delta, x_j^\Delta)$, $h_j^\Delta = x_j^\Delta - x_{j-1}^\Delta$, $j = 1, \dots, N(\Delta)$, $h(\Delta) = \max_j h_j^\Delta$.

The x_j^Δ will be called nodal points, the I_j^Δ elements. $S^p(\Delta)$ denotes the subspace of $\overset{\circ}{H}^1(I)$ consisting of functions that are polynomials of degree $\leq p$ on each element I_j^Δ . The finite element solution $u(\Delta) \in S^p(\Delta)$ is defined by

$$(2.4) \quad B(u(\Delta), v) = B(u, v) = \int_0^1 f v dx \quad \forall v \in S^p(\Delta),$$

i.e., $u(\Delta)$ is the elliptic projection of u onto $S^p(\Delta)$. The expression $e(\Delta) = u - u(\Delta)$ is the error of the finite element solution. Define $\|v\|_E$ to be the energy norm of v :

$$\|v\|_E = [B(v, v)]^{1/2}.$$

We are interested in effectively computing $u(\Delta)$ so that $\|e(\Delta)\|_E$, or the relative error $\|e(\Delta)\|_E / \|u\|_E$, is smaller than some given tolerance.

For any subinterval $I \subseteq I$, let $R(I-I)$ denote the elliptic projection of $\overset{\circ}{H}^1(I)$ onto the space

$$\{v \in \overset{\circ}{H}^1(I) : v(x) = 0 \quad \forall x \in I-I\}$$

and define the error indicators $\eta_j^\Delta(I_j^\Delta) = \eta_j^\Delta$, $j = 1, \dots, N(\Delta)$ and the error estimator $\varepsilon(\Delta)$ as follows:

$$(2.5) \quad \eta_j^\Delta = \|R(I-I_j)e(\Delta)\|_E$$

$$(2.6) \quad \varepsilon(\Delta) = \left[\sum_{j=1}^{N(\Delta)} (\eta_j^\Delta)^2 \right]^{1/2}.$$

The error indicators can be estimated by expressions involving the residuals $r_j = (f - Lu(\Delta))|_{I_j^\Delta}$, see e.g. [8].

We underline the fact that the error indicators and the estimators are directly related to the goal of the computation, in this case to achieve the desired accuracy measured in the energy norm.

An important result is

Theorem 2.1. There exist $C < \infty$ depending on α_1, α_2 of (2.2 a,b) but independent of a, b, f and Δ such that

$$(2.7) \quad \epsilon(\Delta) \leq \|e(\Delta)\|_E \leq C\epsilon(\Delta).$$

If we make additional regularity assumptions about a, b, f then it is possible to prove that the effectivity index

$$(2.8) \quad \theta(\Delta) = \frac{\epsilon(\Delta)}{\|e(\Delta)\|_E} \rightarrow 1 \quad \text{as} \quad \|e(\Delta)\|_E \rightarrow \infty,$$

for more details, see [8].

Let us remark that $\epsilon(\Delta)$ is not in general an upper bound for the error (because $C > 1$). It is possible to estimate C in (2.7), but the estimate we have is very pessimistic. It is also possible to find a computationally more expensive estimator, that provides an upper bound (cf. [8]).

Let us now define a feedback finite element method. In very general terms it consists of consecutive refinements of the meshes in dependence on computed results. More precisely in our model problem we consider only binary meshes, i.e., the nodal points are of the form $j2^{-k}$, and the transition operator performs a bisection of certain elements. By successive application of the transition operator a sequence of meshes is created and the corresponding finite element solutions are computed, until the process is terminated through a stopping criterion.

The sequence of binary meshes $\Delta_1, \Delta_2, \dots$ (with $S^P(\Delta_1) \subseteq S^P(\Delta_{i+1})$) together with the $u(\Delta_1)$ is called a trajectory. A feedback finite element method creates a trajectory which depends on the exact solution u , the coefficients a, b and the initial mesh Δ_1 . Given a performance measure on the set of trajectories a feedback finite element method will be called optimal or adaptive relative to this measure and the set H if for any $(u, a, b, \Delta_1) \in H$ it creates a trajectory with optimal performance measure value (see [11]).

Below we define three simple transition operators $A^{[1]}, A^{[2]}, A^{[3]}$; in each case we list the elements \sum_1 that are bisected in passing from Δ_1 to Δ_{i+1} :

$$\sum_1^{[1]} : \text{all elements } I_j^{\Delta_1} \text{ for which } \eta_j^{\Delta_1} \geq \beta \max_{I \in \Delta_1} \eta^{\Delta_1}(I) \quad 0 \leq \beta \leq 1$$

$$\sum_i^{[2]} : I_{j(1)}^{\Delta_1}, \dots, I_{j(k)}^{\Delta_1}, \text{ where } \eta_{j(1)}^{\Delta_1} \geq \eta_{j(2)}^{\Delta_1} \geq \dots \geq \eta_{j(N(\Delta_1))}^{\Delta_1}$$

in an ordering (by magnitude) of the indicators and

$$k = [\gamma N(\Delta_1)] + 1 \quad 0 \leq \gamma < 1$$

([a] denotes the integer part of a).

$$\sum_1^{[3]} : \sum_1^{[1]} \cup \sum_1^{[2]}.$$

The transition operator $A^{[1]}$ bisects all elements whose indicators are larger than or equal to β times the largest indicator; for $\beta = 0$ this implies bisection of all elements. $A^{[2]}$ bisects approximately the fraction γ of the elements and the transition operator $A^{[3]}$ bisects all elements with indicators larger than or equal to β times the largest one, and at least the fraction γ .

In practice it is advantageous to use a more composite transition operator. Define the predictor $\tilde{\eta}(I_j^{\Delta_1})$ by

$$\tilde{\eta}(I_j^{\Delta_1}) = \min \left(\frac{[\eta^{\Delta_1}(I_j^{\Delta_1})]^2}{\eta^{\Delta_k}(I_j^{\Delta_k})}, \quad \kappa \eta^{\Delta_1}(I_j^{\Delta_1}) \right), \quad 0 < \kappa < 1$$

where $I_j^{\Delta_k} \in \Delta_k$, $k < 1$, is the direct predecessor of $I_j^{\Delta_1}$. The direct predecessor of $I_j^{\Delta_1}$ is defined as the $I_j^{\Delta_k}$ (with maximal k) that satisfies $I_j^{\Delta_k} \supseteq I_j^{\Delta_1}$, $\eta_j^{\Delta_k} = 2\eta_j^{\Delta_1}$, except for $I_j^{\Delta_1} \in \Delta_1$ where the direct predecessor is defined to be $I_j^{\Delta_1}$. $\tilde{\eta}$ is called a predictor since it uses past experience to predict the effect of bisection on the value of the indicator. The transition operator $A^{[4]}$ bisects all elements $I_j^{\Delta_1}$ that satisfy

$$\eta^{\Delta_1}(I_j^{\Delta_1}) \geq \beta \max_{I \in \Delta_1} \tilde{\eta}(I) \quad 0 \leq \beta \leq 1.$$

The transition operator, $A^{[5]}$, used in the FEARS program is a direct extension of the operator $A^{[4]}$, including the so-called short passes: whenever the increase in the number of elements is less than a preset percentage of the total number of element of Δ_1 no finite element solution is computed on Δ_{i+1} . Instead indicators for the elements of Δ_{i+1} are constructed directly from the $\tilde{\eta}(I_j^{\Delta_1})$ and the operator $A^{[4]}$ is applied to $\Delta_1, \dots, \Delta_i, \Delta_{i+1}$ with this set of indicators for $\eta^{\Delta_{i+1}}(I_j^{\Delta_{i+1}})$. The process continues until the increase in the number of elements is larger than the preset percentage of the total number of elements of Δ_1 , at which point the short pass is concluded and the finite element solution is computed on the resulting mesh Δ_{i+k} .

The transition operators $A^{[1]} - A^{[5]}$ all have the property that they bisect at least one element with $\eta_j^{\Delta_1} \geq \beta \max_{I \in \Delta_1} \eta^{\Delta_1}(I)$, $0 < \beta \leq 1$ (indeed, they all bisect the one with the largest indicator). We shall call a transition operator, and the corresponding feedback finite element method, with this property regular. The following theorem is proven in [9],

Theorem 2.2. Let $u \in H^1(I)$ denote the solution of (2.1 a,b) and let $u(\Delta_i)$ $i = 1, 2, \dots$ be the sequence of solutions created by a regular feedback finite element method, then

$$\|e(\Delta_i)\|_E \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

We note that $h(\Delta_i)$ does not necessarily converge to zero.

If we define the performance measure (of a trajectory) to be 1 whenever $u(\Delta_i)$ converges to u (in the energy norm) and 0 otherwise, then Theorem 2.1 simply states that any regular feedback finite element method is adaptive with respect to this measure and the set of $u \in H^1(I)$, a, b satisfying (2.2 a)-(2.2 c) and arbitrary Δ_1 .

Let ϕ be defined as follows

$$\phi(u, N) = \inf_{\substack{\Lambda: \text{binary mesh} \\ N(\Lambda) = N}} \inf_{v \in S^P(\Lambda)} \|u - v\|_E$$

and consider the performance measure that is 1 whenever

$$(2.9) \quad \|u - u(\Delta_1)\|_E \leq C\phi(u, N(\Delta_1))$$

(for some C independent of 1), and 0 otherwise.

In [9] we analyze the adaptivity with respect to this measure in the case when a in addition to satisfy (2.2 a) is Hölder continuous with exponent $\alpha = 1/2$. The transition operators $A^{[1]} - A^{[5]}$ are not adaptive with respect to this measure for all $u \in \dot{H}^1(I)$. However, it is possible to verify adaptivity for a rather large class of function $K \subseteq \dot{H}^1(I)$. We shall not give here the exact definition of K , instead we refer to [9]. The class K contains for instance all functions that are smooth or have singular behaviour as x^α , $\alpha > 1/2$ ($x^\alpha - x \in K$). The following theorem is proven in [9]:

Theorem 2.3. Assume that a is Hölder continuous with exponent $1/2$ and a, b satisfy (2.2 a,b). Let Δ_1 be a sequence of meshes generated by a feedback finite element method based on either of the transition operators $A^{[1]}$ or $A^{[4]}$ with $0 < \beta$; then

$$\|e(\Delta_1)\|_E \leq C(u)\phi(u, N(\Delta_1))$$

provided $u \in K$.

Theorem 2.4. Let the assumption be as in the previous theorem. Let Δ_1 be a sequence of meshes generated by a feedback finite element method based on either of the transition operators $A^{[2]}$ or $A^{[3]}$ with $\beta > 0$.

For any $u \in K$ there exists γ_0 such that if $0 \leq \gamma \leq \gamma_0$, then

$$\|e(\Delta_1)\|_E \leq C(u)\phi(u, N(\Delta_1)).$$

It should be noted that $A^{[1]}$ and $A^{[4]}$ with $\beta = 0$ do not in general (e.g. when $u = x^\alpha - x$) produce optimally convergent finite element solutions; consequently $A^{[1]}$ and $A^{[4]}$, with $\beta = 0$, are not adaptive relative to K (they are adaptive relative to a subclass of smooth functions $\subseteq K$).

We also observe that in most practical cases $C_1 N^{-p} \leq \phi(u, N) \leq C_2 N^{-p}$, whether u is smooth or not (e.g. for $u = x^\alpha - x$, $\alpha > 1/2$). Thus a feedback finite element method which is adaptive with respect to this performance measure leads to a rate of convergence that (in

practical cases) is not influenced by the smoothness of the solution.

Let the cost of computing the finite element solution $u(\Delta_i)$, the indicators $\eta_j^{\Delta_i}$, the estimator $\varepsilon(\Delta_i)$ and creating the mesh Δ_{i+1} be denoted by $\varphi(\Delta_i)$. The total cost $\psi(\Delta_j) = \sum_{i=1}^j \varphi(\Delta_i)$ and the proportionality factor

$$\xi(\Delta_i) = \frac{\psi(\Delta_i)}{\varphi(\Delta_i)}$$

describes the effectivity of the feedback method. Assuming that $\varphi(\Delta_i) = N^\lambda(\Delta_i)$, $\lambda \geq 1$, then for the case of the operators $A^{[2]}$ and $A^{[3]}$ we have

$$N(\Delta_i) \leq \frac{1}{1+\gamma} N(\Delta_{i+1}),$$

and hence

$$\begin{aligned} \psi(\Delta_j) &\leq \sum_{i=0}^{j-1} \left(\frac{1}{(1+\gamma)^\lambda} \right)^i N^\lambda(\Delta_j) \\ &\leq \frac{(1+\gamma)^\lambda}{(1+\gamma)^\lambda - 1} N^\lambda(\Delta_j). \end{aligned}$$

Thus for the operators $A^{[2]}$ and $A^{[3]}$ we get

$$\xi(\Delta_i) \leq \frac{(1+\gamma)^\lambda}{(1+\gamma)^\lambda - 1}.$$

A similar inequality is not (in general) true for $A^{[1]}$ or $A^{[4]}$; whenever only very few new elements are added in each transition the ratio $\xi(\Delta_i)$ becomes unbounded as $i \rightarrow \infty$. The operator $A^{[5]}$ is created exactly in order to avoid this difficulty.

In the preceding analysis we considered only the case of one given load function f , but the generalization to multiple loads is straightforward: We interpret the multiple load problem as a system of equations for u_ℓ , $1 \leq \ell \leq n$. We introduce the energy norm

$$\left[\sum_{\ell=1}^n \|u_{\ell} - u_{\ell}(\Delta_1)\|_E^2 \right]^{1/2}$$

and we define the indicators and estimators by the appropriate projections in this norm.

3. Two Dimensional Problems: The FEARS Program. Many of the ideas discussed for the simple one dimensional problem of the previous section have been implemented in a two dimensional setting in the FEARS program. FEARS is able to handle symmetric elliptic systems of two equations on regions which are unions of curvilinear quadrilaterals. Within the program, each of these quadrilaterals is mapped onto a unit square. The actual finite element calculations are carried out on these mapped squares using conforming square bilinear elements. A typical FEARS mesh for the shaded region of Fig. 1 is shown in Fig. 2. In this paper we shall illustrate some of the major features of the program by reference to a model problem and by analogy with the one dimensional case discussed in Section 2. For further details and a fuller discussion see [1], [5], [7] and [10].

As the model problem, consider the stress state of the cracked panel illustrated in Figure 1. We suppose that plane strain assumptions apply with ν = Poisson's ratio = .3 and E = Young's modulus = 1. (Because of the symmetries which are present, we need only compute on the shaded quarter panel.) As in Section 2, we shall be interested in judging the accuracy of our finite element approximation in terms of the strain energy norm of the error. The exact solution of this model problem has an $r^{1/2}$ -type singularity in displacements near the crack tip, as well as less severe singular behaviour at the corners. Indeed, the solution is in the Besov space $B_{2,\infty}^{\mu}$ for $\mu = 3/2$ (and no greater). As a consequence, using uniform meshes we would only expect an $O(N^{-1/4})$ rate of convergence in the energy norm. Better convergence rates than this can be achieved by appropriate mesh refinements.

As in the one dimensional case, error indicators can be associated with each element of the mesh. However, the computation of such an indicator now needs information not only about the finite element solution on the element itself, but also on adjacent elements. Error estimators are constructed from these indicators exactly as in Section 2.

The transition operators used are direct analogues of $A^{[4]} - A^{[5]}$. Theorems 2.1 and 2.2 carry over directly to the present case, while (2.8) holds under some additional assumptions on the meshes. Experiments seem to indicate that these extra assumptions are necessary in the two dimensional case, and that the analogues of Theorems 2.3 and 2.4 appear to hold.

In Figure 3 we have plotted the relative error in the energy norm against the number of degrees-of-freedom N for two feedback trajectories produced by FEARS: I was created using the transition operator

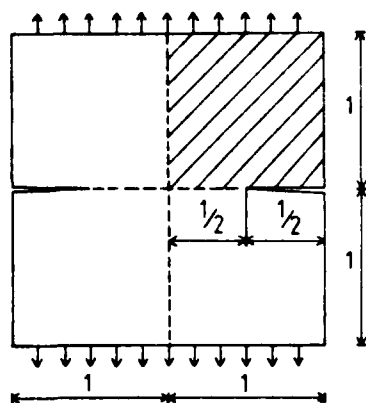


Figure 1

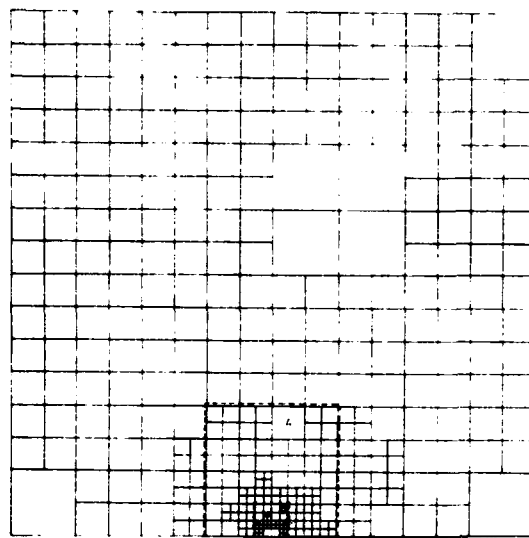


Figure 2a

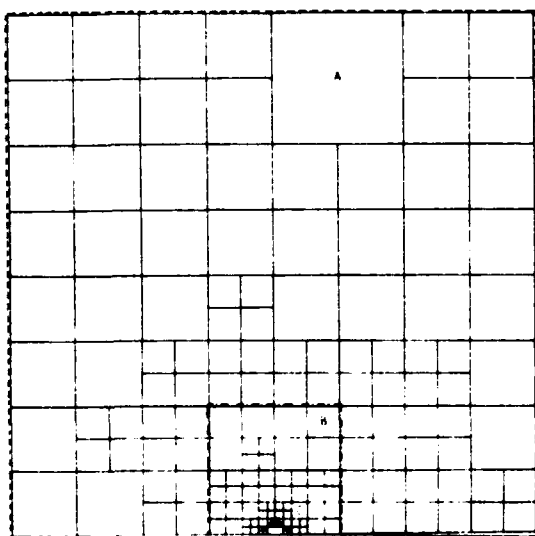


Figure 2b

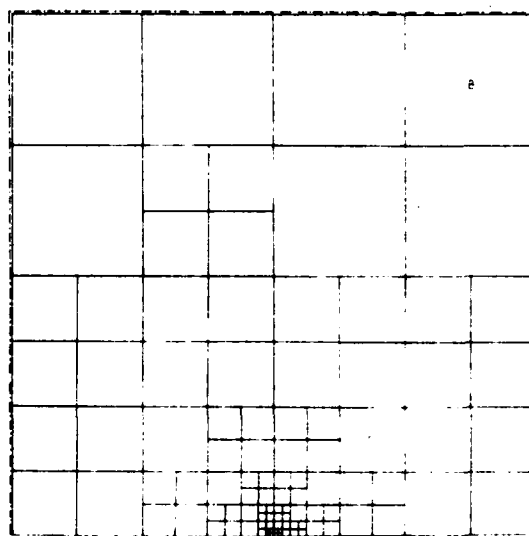


Figure 2c

$A^{[5]}$ with $\beta = 1$ and $\gamma = .2$; II used $A^{[5]}$ with $\beta = 0$ (which leads to a uniform mesh). As theory would predict the trajectory II of uniform meshes shows an $O(N^{-1/4})$ rate of convergence. However I shows an $O(N^{-1/2})$ rate. This is the maximum possible rate for elements of degree 1. The mesh shown in Fig. 2 corresponds to the last computed mesh of I, and, as expected, it exhibits a severe refinement around the crack tip. The ratio of maximum element size to minimum element size for this mesh is 256.

Rather more meaningful than the dependence of relative error on N is its relation to total cost (i.e. computer time). Fig. 4 is a plot of accuracy against computer time for the trajectories I and II, and for the trajectory III which was created by $A^{[4]}$ with $\beta = 1$ i.e. no "short passes" were permitted. Again we see the marked superiority of I over II. Also, for a given accuracy, I requires approximately half as much time as III. The trajectory III "wastes" time solving on meshes that only differ slightly from one another. The proportionality factors for trajectory I are in the range $1.5 < \xi < 2.5$, while those for III lie in the range $3 < \xi < 5$. For problems with more severe singular behaviour than our model problem (e.g. a clamped-free crack) the relative economy of I over III could be expected to be even greater.

Finally in Figure 5 we show the dependence of the effectivity index θ on solution accuracy. The plot is consistent with $\theta \rightarrow 1$ as $\|e\|_E \rightarrow 0$. Let us just mention that FEARS has two kinds of error indicators and estimators. Asymptotically both behave similarly, though one is always greater than the other. In our calculations we used the larger. This is usually preferable since it provides extra "safety" in the stopping criterion.

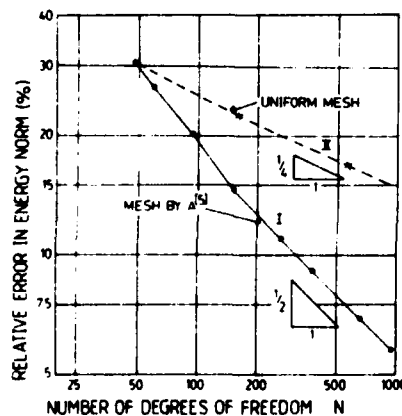


Figure 3

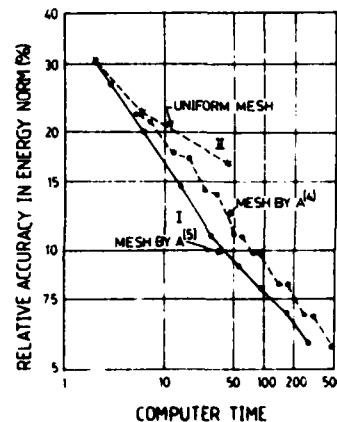


Figure 4

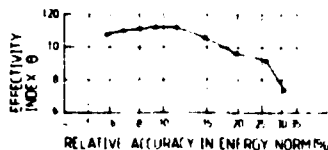


Figure 5

4. Post-Processing Calculations. So far we have only been judging the accuracy of a finite element solution in terms of the energy norm $\|u - u(\Delta)\|_E$ of the finite element error. However, in practice, often the goal of a finite element calculation is not to obtain high accuracy in the energy norm as such, but rather to find sufficiently accurate approximations for values of certain functionals (for instance, stress at a point, stress intensity factor at a crack tip, average flow rate through a surface, etc.). In this section we shall address this point by reference to a model problem. For a more complete discussion see [2], [3] and [4].

Let us consider a stress function formulation of the torsion problem for a square bar. The governing boundary value problem is

$$(4.1a) \quad \nabla^2 u = -1 \quad \text{on } \Omega = (-1,1) \times (-1,1)$$

$$(4.1b) \quad u = 0 \quad \text{on the boundary } \partial\Omega \text{ of } \Omega.$$

We shall be interested in this problem only as a means of finding the (maximum) stress

$$(4.2) \quad \sigma = \left. \frac{\partial u}{\partial x_1} \right|_{x=(1,0)}.$$

Given the finite element approximation $u(\Delta)$, the most common approach would be to use $\left. \frac{\partial u(\Delta)}{\partial x_1} \right|_{x=(1,0)}$ as an approximation to σ . However,

we shall employ a post-processing technique which yields better results than this standard pointwise evaluation approach. It may be shown that for the problem (4.1)

$$(4.3) \quad \left. \frac{\partial u}{\partial x_1} \right|_{x=(1,0)} = \int_{\Omega} (u \xi + \psi) dA,$$

where

$$(4.4a) \quad \xi = -\nabla^2 \psi_0$$

$$(4.4b) \quad \varphi = \frac{1}{\pi} \frac{x_1 - 1}{(x_1 - 1)^2 + x_2^2} - \varphi_0$$

and φ_0 is any smooth function which ensures that $\varphi = 0$ on $\partial\Omega - (1,0)$. For instance, we could take

$$(4.4c) \quad \varphi_0 = \frac{1}{\pi} \left(\frac{x_1 - 1}{(x_1 - 1)^2 + 1} - \frac{x_1 - 1}{5} + \frac{x_1 - 1}{x_2^2 + 4} \right).$$

The relation (4.3) can readily be verified by an integration by parts and a simple limiting argument (see [2]). Obviously (4.3) suggest that we use

$$(4.5) \quad \sigma(\Delta) = \int_{\Omega} (u(\Delta)\xi + \varphi) dA$$

as an approximation for σ . The expression (4.5) is one example of an extraction expression for σ , and is the basis of the post-processing calculation that we shall use. Let us just mention that the standard pointwise evaluation approach can also be written in the form (4.5) if

we formally let $\varphi_0 = \frac{1}{\pi} \frac{x_1 - 1}{(x_1 - 1)^2 + x_2^2}$. (So $\varphi = 0$ and ξ is a derivative of the Dirac delta function.)

Let us now discuss the accuracy of the post-processed value $\sigma(\Delta)$. Clearly

$$(4.6) \quad \sigma - \sigma(\Delta) = \int_{\Omega} (u - u(\Delta))\xi dA,$$

and if we now introduce an auxiliary problem

$$(4.7) \quad \begin{aligned} \nabla^2 \psi &= -\xi, & \text{on } \Omega \\ \psi &= 0 & \text{on } \partial\Omega \end{aligned}$$

then (4.6) gives

$$(4.8) \quad \sigma - \sigma(\Delta) = \int_{\Omega} \nabla(u - u(\Delta)) \cdot \nabla(\psi - \psi(\Delta)) dA.$$

This shows that the accuracy of $\sigma(\Delta)$ depends not only on the accuracy of $u(\cdot)$ but also on the error $(\psi - \psi(\Delta))$. Clearly the choice of finite element mesh should recognize this fact.

There are some alternate forms of (4.8):

$$(4.9) \quad \sigma - \sigma(\Delta) = \frac{1}{4} (\|(u + \psi) - (u + \psi)(\Delta)\|_E^2 - \|(u - \psi) - (u - \psi)(\Delta)\|_E^2)$$

and somewhat less sharply

$$(4.10a) \quad |\sigma - \sigma(\Delta)| \leq \|u - u(\Delta)\|_E \|\psi - \psi(\Delta)\|_E$$

$$(4.10b) \quad |\sigma - \sigma(\Delta)| \leq \frac{1}{2\alpha} (\|u - u(\Delta)\|_E^2 + \alpha^2 \|\psi - \psi(\Delta)\|_E^2),$$

where for any function v we have used $\|v\|_E = (\int_{\Omega} |\nabla v|^2 dA)^{1/2}$ to denote the energy norm associated with (4.1). Let us mention that (4.10a) and (4.10b) in general give upper bounds for the error in $\sigma(\Delta)$ as they fail to take account of any cancellation in the integral (4.8).

The feedback finite element method already discussed in Sections 2 and 3 can be applied to this post-processing calculation provided the appropriate performance measures are chosen. It is possible to prove that Theorem 2.2 (relating to the convergence measure) holds if the feedback method is regular with respect to the indicators associated with the norm $(\|u\|_E^2 + \alpha^2 \|\psi\|_E^2)^{1/2}$. (See the remarks on multiple load computations at the end of Section 2.) However, it seems too much to ask that the direct analogues of Theorem 2.3 and 2.4 should hold. Because of the possibility of cancellation in the integral (4.8) we can expect that

$$\phi(N) = \inf_{\substack{\Delta \\ N(\Delta)=N}} |\sigma - \sigma(\Delta)|$$

could be quite small (possibly zero), even for small values of N . However, we can frame a less demanding performance measure based on (4.10b). Let

$$\phi(N) = \inf_{\substack{\Delta \\ N(\Delta)=N}} \|u - u(\Delta)\|_E^2 + \alpha^2 \|\psi - \psi(\Delta)\|_E^2.$$

The analogues of Theorems 2.3 and 2.4 are now closely related to the corresponding results in Section 3.

A posteriori estimates for this post-processing calculation may be based on (4.9) and (4.10). FEARS gives estimators $\epsilon_1(\Delta)$, $\epsilon_2(\Delta)$ for $\|(u + \psi) - (u + \psi)(\Delta)\|_E$ and $\|(u - \psi) - (u - \psi)(\Delta)\|_E$ respectively with effectivity indices which converge to 1. Now taking the obvious arithmetic combinations of these estimators suggested by (4.9),

$$(4.11) \quad \varepsilon(\Delta) = \frac{1}{4}(\varepsilon_1(\Delta)^2 - \varepsilon_2(\Delta)^2),$$

gives an estimator for the error $\sigma - \sigma(\Delta)$ in the post-processed value. This estimator will have an effectivity index which converges to 1, provided the subtraction in (4.11) does not lead to significant cancellation. If this critical case occurs, we cannot expect $\varepsilon(\Delta)$ to be reliable. We may nonetheless use estimators based on (4.10). Such estimators however will be pessimistic, especially in this critical case. See [4] for further discussion of this point.

Table 1 shows some numerical results for our model problem. The problems (4.1) and (4.7) were run as a multiple load system using FEARS. Since both solutions u and ψ are rather smooth it is not surprising that, at least initially, all the meshes produced were uniform. Notice the following features of the results in Table 1:

(a) $\|u - u(\Delta)\|_E$ displays the usual $O(h)$ rate of convergence expected of bilinear elements for a smooth exact solution u .

(b) The effectivity index of the estimator for $\|u - u(\Delta)\|_E$ is close to 1.

(c) The post-processed value $\sigma(\Delta)$ is markedly superior to $\frac{\partial u(\Delta)}{\partial x_1} \Big|_{x=(1,0)}$. The standard value is converging as $O(h)$, whereas the convergence rate of the post-processed value is $O(h^2)$.

(d) The effectivity index for the estimator (4.11) is very close to 1, whereas for the estimator based on (4.10a) it seems to stabilize around 1.3. The fact that (4.10a) leads to an estimator which is only a slight overestimate of $|\sigma - \sigma(\Delta)|$ indicates that there is little cancellation in the integral (4.8). The high accuracy in $\sigma(\Delta)$ can be explained solely by (4.10a). Indeed, for this problem, since ψ is smooth we expect $\|\psi - \psi(\Delta)\|_E = O(h)$. Thus (4.10a) gives $|\sigma - \sigma(\Delta)| = O(h^2)$, which is consistent with (c) above.

TABLE 1

Accuracy of the maximum stresses calculated by post-processing,
and a posteriori error estimates.

NUMBER OF ELEMENTS (Uniform mesh, h = element size)	16 ($h=.5$)	64 ($h=.25$)	256 ($h=.125$)
ENERGY NORM OF THE ERROR $\ u - u(\Delta)\ _E / \ u\ _E$	30.1%	15.2%	7.62%
EFFECTIVITY INDEX OF ESTIMATOR FOR $\ u - u(\Delta)\ _E$.94	.96	.98
ERROR IN STANDARD VALUE $\left \left(\sigma - \frac{\partial u(\Delta)}{\partial x_1} \right)_{x=(1,0)} \right / \sigma $	29%	16%	8.7%
ERROR IN POST-PROCESSED VALUE $ \sigma - \sigma(\Delta) / \sigma $	1.5%	.37%	.089%
EFFECTIVITY INDEX OF THE ESTIMATOR BASED ON (4.11)	.99	1.02	1.01
(4.10a)	1.24	1.28	1.29

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